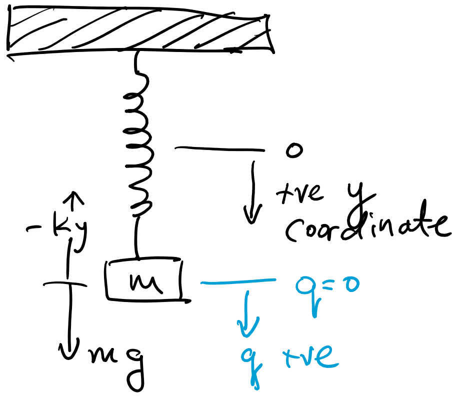
Symplectic Integrators

Monday, 23 October 2023 11:05

ODE → Runge-Kutta Methods "Black Box"

Harmonic Oscillator



$F = mg - ky$

to make things easier

$m = 1 \quad k = 1$

$F = g - y$

$\boxed{F = ma}$

Define $q = 0$ where $F = 0$

$q = y - g = -F$

Newton's Law $F = ma = m\frac{d^2y}{dt^2} = m\ddot{y}$

$\ddot{q} = \ddot{y} = F$

$\ddot{q} = -q$ ← 2nd order ODE

- to get to 2 1st order ODEs we introduce p

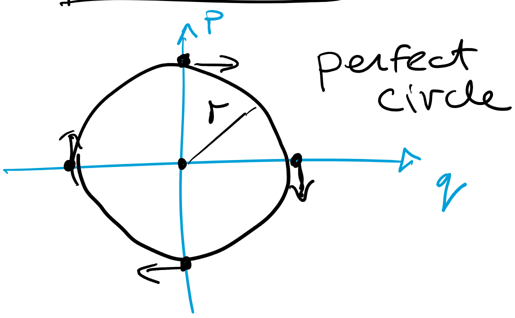
Momentum

$p = mv = m\dot{y} = \dot{q}$

$p = \dot{q}$

$$\boxed{\begin{aligned} \dot{p} &= -q \\ \dot{q} &= p \end{aligned}}$$

Harmonic Oscillator



perfect circle

$r^2 = q^2 + p^2$

equation for circle

"Energy" $= q^2 + p^2$

conserved for all time

$$\left.\begin{aligned} q_{n+1} &= q_n + h p_n \\ p_{n+1} &= p_n - h q_n \end{aligned}\right\}$$ Equations for steps in the Forward Euler

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$$q_{n+1}^2 + p_{n+1}^2 = (1+h^2)(q_n^2 + p_n^2)$$

$$r_{n+1}^2 = (1+h^2)\, r_n^2$$

Exponential growth of the radius! "Energy"

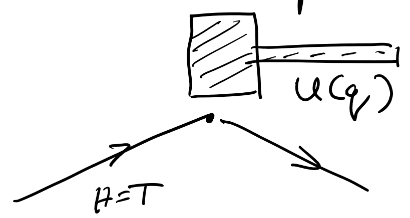
Not periodic

$$E = \underset{\text{Kinentic}}{T} + \underset{\text{Potential}}{U} = \frac{1}{2}mv^2 + \frac{1}{2}ky^2$$
$$= \frac{1}{2}(p^2 + q^2)$$

$$H(p,q) = \frac{1}{2}(p^2+q^2) = T(p) + U(q)$$

Seperable

$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}}$$



$T(p) \longrightarrow$ "free motion" DRIFT

$U(q) \longrightarrow$ only changes the momentum "Impulse" KICK

Leapfrog Method, Strömer-Verlet

For the Harmonic Oscillator

$$h = \Delta t \begin{cases} q_{n+1/2} = q_n + \frac{1}{2} h p_n & \text{"half drift"} \\ p_{n+1} = p_n - h\, q_{n+\frac{1}{2}} & \text{"full kick"} \\ q_{n+1} = q_{n+\frac{1}{2}} + \frac{1}{2} h p_{n+1} & \text{"half drift"} \end{cases}$$

In terms of $\underline{x}$ and $\underline{v}$

$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

$$x_{n+\frac{1}{2}} = x_n + \frac{1}{2} h\, v_n$$

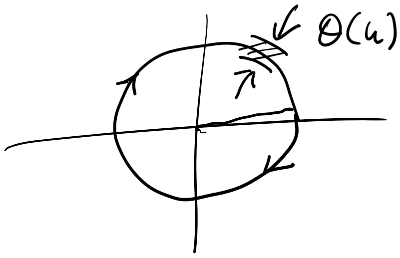
In terms of $\underline{x}$ and $\underline{v}$

$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

$$\underline{x}_{\frac{1}{2}} = \underline{x}_0 + \tfrac{1}{2} h \underline{v}_0$$

$$\underline{v}_1 = \underline{v}_0 + h\left(-\nabla U(\underline{x}_{\frac{1}{2}})\right)$$

$$\underline{x}_1 = \underline{x}_{\frac{1}{2}} + \tfrac{1}{2} h \underline{v}_1$$



There is an error

$O(h^3)$ local error!

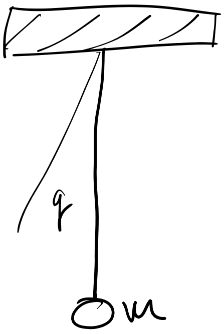
Error is in the Angle!

Slight error in the Period.

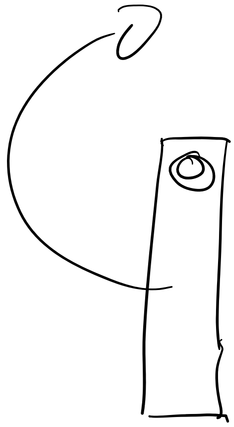
$$\widetilde{\text{Period}} \neq \text{Period}$$

numerical

$$\tilde{H} \neq H$$



|| Foucault Pendulum



Simple Pendulum

$$H = \frac{1}{2}p^2 - \varepsilon \cos q$$