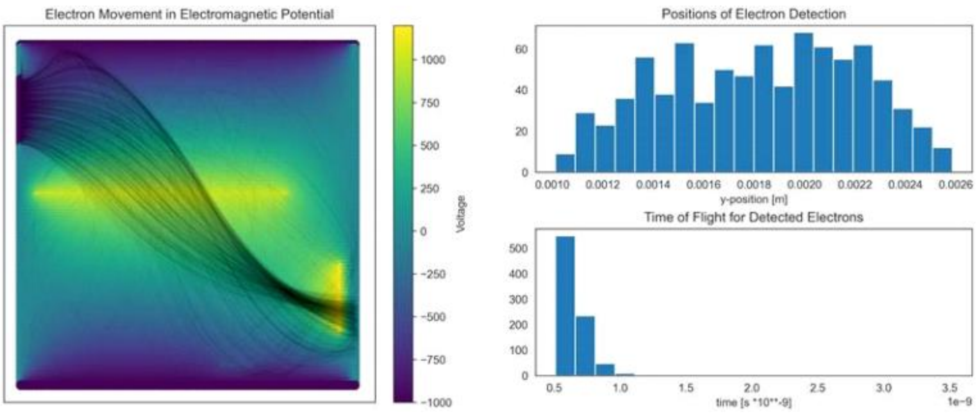
1-D Hydrodynamics

Monday, 4 December 2023 10:56

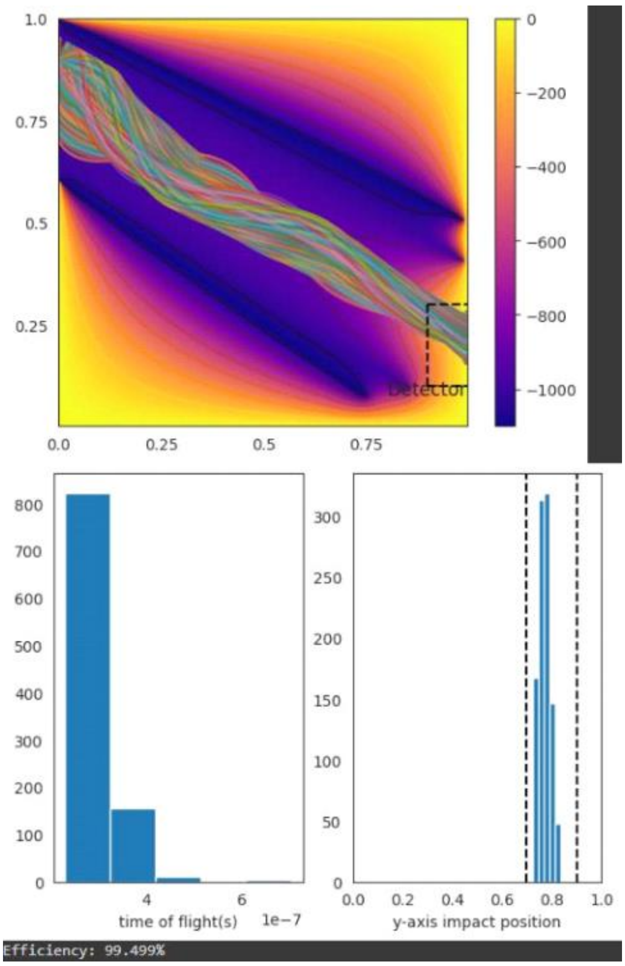
Design Prize:

Honorable Mentions:

Rafaela Lambrinos Maria: for a functional and interesting design!

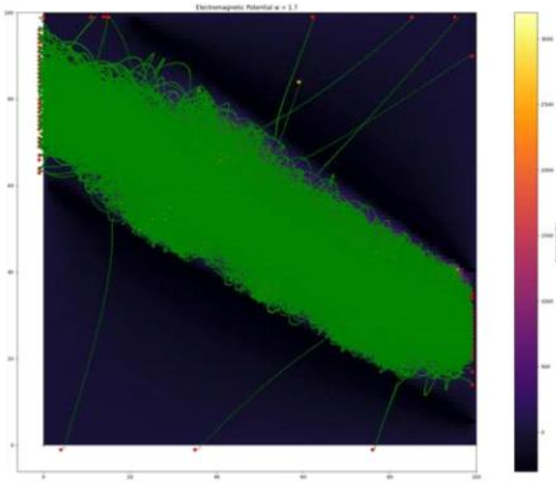


Luigi Willi Pietro: Bi-cubic interpolation! (attempt) Very high detection rate with a funnel-like design.

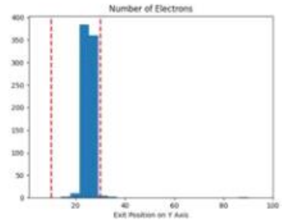


Winner:

Timo Elias Rieger: Best documented design, with plenty of explorations!




```
remained inside:  151  ,ratio:  15.1 %
worng exit:  93  ,ratio:  9.3 %
correct exit:  756  ,ratio:  75.6 %
```



Recall : LAX Method

$$\rho_i^{(n+1)} = \frac{1}{2}\left(\rho_{i+1}^{(n)} + \rho_{i-1}^{(n)}\right) - \frac{c}{2}\left(\rho_{i+1}^{(n)} - \rho_{i-1}^{(n)}\right)$$

$$c = \frac{\Delta t\, u}{\Delta x}$$

For linear advection $f(\rho) = \rho u$

Rewrite the above in terms of fluxes

$$\boxed{A} \quad \rho_i^{(n+1)} = \frac{1}{2}\left(\rho_{i+1}^{(n)} + \rho_{i-1}^{(n)}\right) - \frac{\Delta t}{2\Delta x}\left(F_{i+1}^{(n)} - F_{i-1}^{(n)}\right)$$

also Rewrite the 1^{ST} order upwind method:

$$\boxed{A'} \quad \rho_i^{(n+1)} = \rho_i^{(n)} - \frac{\Delta t}{\Delta x}\left(F_i^{(n)} - F_{i-1}^{(n)}\right) \quad \text{for } u \geq 0$$

$$- \frac{\Delta t}{\Delta x}\left(F_{i+1}^{(n)} - F_i^{(n)}\right) \quad \text{for } u < 0$$

Now for 1-D hydrodynamics, for 3 conserved quantities:

1. Cons. of Mass : $F = \rho u$ — Note u doesn't have to be a constant here.
2. Cons. of Momentum: $F = \rho u^2 + P$ — pressure
3. Cons. of Energy: $F = u(E+P)$

Consider a state vector

$$\underline{U} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} \quad F(U) = \left[\rho u,\ \rho u^2 + P,\ u(E+P)\right]$$

along diagonal

\E/

$$\boxed{\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0}$$

very general equation of "stuff"

3 equations

(in our 1-D case we have $\nabla \cdot \equiv \frac{\partial}{\partial x}$)

Unknowns: $\rho, \rho u, E, P$

4 unknowns ...

Equation of State

$$E = \text{Kinetic Energy} + \text{Thermal Energy}$$

$$\frac{1}{2}\rho u^2 + \rho e$$

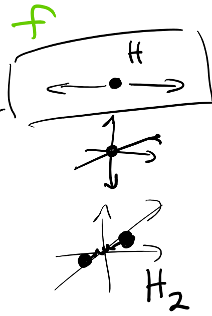
e: thermal energy density per unit mass (specific)

$$\rho, \rho u, e, P$$

$$PV = nRT \quad \text{(Ideal Gas)}$$

(same) $$e = \frac{P}{(\gamma-1)\rho} \qquad \gamma = \frac{f+2}{f}$$

f: degrees of freedom.



$f=1$

$f=3$

3-translation
2-rot
1-vibrational
6

$$\gamma = 3$$

$$e = \frac{1}{2}\frac{P}{\rho}$$

$$E = \frac{1}{2}\rho u^2 + \frac{1}{2}P$$

$$F(U) = \left[\rho u,\ \rho u^2 + P,\ u\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)\right]$$

Think about Finite Volumes:

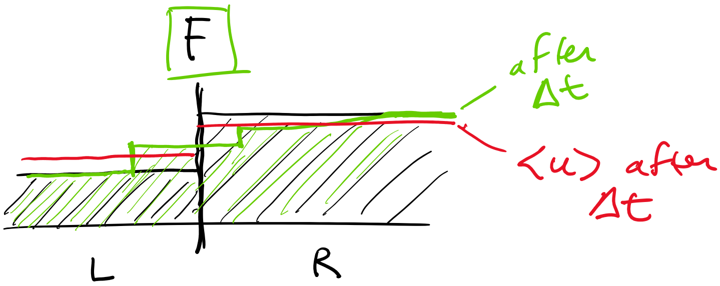
U as representing the average value of the "material" in the cell:

$$\langle U \rangle_i^n = \frac{1}{\Delta x}\int^{x_{i+\frac{1}{2}}} U(x, t^n)\, dx$$

$$\langle u \rangle_i^n = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}} u(x, t^n)\, dx$$

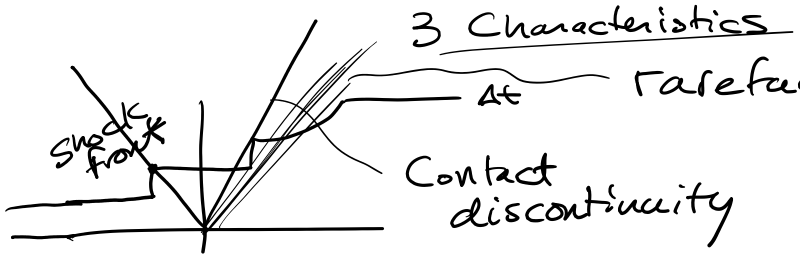
in actual fact we should consider this average over a timestep.

$$\frac{\langle u \rangle_i^{n+1} - \langle u \rangle_i^n}{\Delta t} + \frac{F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n}{\Delta x} = 0$$



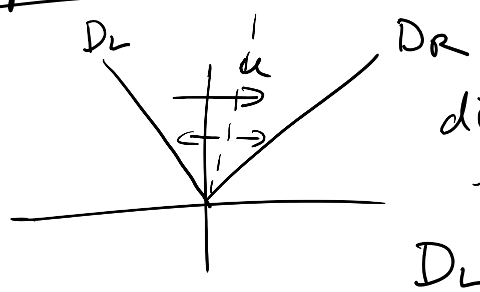
$F(u_L, u_R)$
Riemann Solver

★ Good Approximation



3 equations

Approximate Riemann Solver :



discontinuity speeds of the left and right waves.

$$D_L = u - c_s$$
$$D_R = u + c_s$$

$$D_{max} = \max(D_L, D_R) = |u| + c_s$$

$$c_s = \sqrt{\gamma \frac{P}{\rho}} \quad \text{speed of sound}$$

Lax-Friedrichs Riemann Solver

$$\boxed{F_{i-\frac{1}{2}}^{n} = \frac{1}{2}\left[F_i^n + F_{i-1}^n\right] - \frac{1}{2} D_{max} \left[u_i^n - u_{i-1}^n\right]}$$

Method $\boxed{B}$

$$\langle u \rangle_i^{n+1} = \langle u \rangle_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+\frac{1}{2}}^{n\ (n+\frac{1}{2})} - F_{i-\frac{1}{2}}^{n\ (n+\frac{1}{2})}\right]$$

Method $\boxed{C}$ (drop <>)

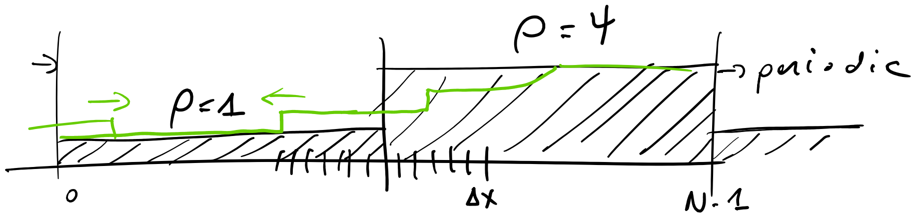
$$\underline{\underline{{}^{*}u_i^{n+\frac{1}{2}}}} = {}^{*}u_i^n - \frac{\Delta t}{2\Delta x}\left[F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right]$$

"predictor step"

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[\underline{F({}^{*}u_i, {}^{*}u_{i+1})} - F({}^{*}u_{i-1}, {}^{*}u_i)\right]$$

"Corrector Step"

use LAX-Friedrichs Riemann Solver

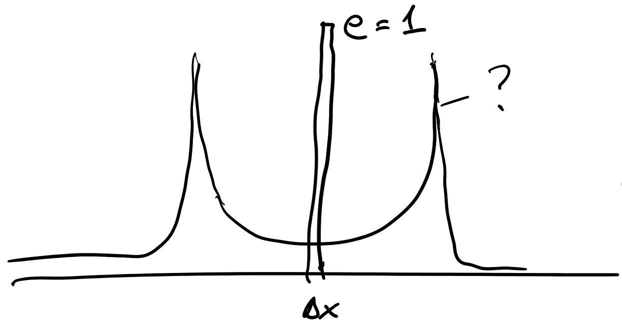


SOD shock-tube problem.

$u = 0$

$e = 10^{-5}$ — solve for P given ρ

Δt ? $\boxed{D_{max} < \frac{\Delta x}{\Delta t}}$



$\rho = 1,\ u = 0$
$e = 10^{-5}$

Sedov-Taylor Blast wave
solution (like a bomb)

Sedov-Taylor Blast wave
solution (like a bomb)